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Resolution of the Catch Equation by a Simple Iterative Method

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Abstract

The solution (F) to the catch equation when catch and population size in numbers are known is determined quickly and accurately using NEWTON's iterative method on a pocket calculator. The method and program are discussed.

Introduction

In an attempt to estimate monthly stock sizes and fishing mortality rates of Northwest Atlantic squid species by cohort analysis, the data series proved inadequate for providing the expected convergence of estimates for these parameters. This provided the opportunity to review the different approaches to the solution of the catch equation, and to test them on artificial cohorts. One of the possible solutions is presented and discussed in this paper.

The Catch Equation

Considering a single cohort (year-class or brood) in successive periods (t) of its exploited phase, the parameters associated with fishing are:

C_t = catch in numbers during period t,

N_t = size of cohort (in numbers) at the beginning of period t,

$Z_t = M + F_t$ is the total instantaneous mortality rate for period t, where M and F_t are the natural and fishing mortality rates respectively.

The catch equation $C_t = \frac{F_t}{F_t+M} N_t(1 - \exp(-F_t-M))$ and the survival equation $N_{t+1} = N_t \exp(-F_t-M)$ can be related by the ratio

$$\frac{N_{t+1}}{C_t} = \frac{(F_t+M)\ \exp(-F_t-M)}{F_t(1 - \exp(-F_t-M))} = r \quad \dots\dots\dots\dots \quad (1)$$

Therefore, if N_{t+1}, C_t and M are known, F_t can be calculated by solving (1) and used in the survival equation for the calculation of N_t. This process can be performed backwards to the first periods of exploitation of the cohort. A starting value of F_T and a corresponding N_T for one of the last periods of fishing must be assumed to start the computation.

The main problem is to solve the complex relationship (1) in F_t. For common values of M, tables of r as a function of F can be used (Schumacher, 1971). Otherwise, numerical resolution can be performed by iterative methods. Doubleday (MS 1975, MS 1977) and Miller (1977) have used the "regula falsi" after linearizing the function of F. The use of Newton's iterative method is not mentioned in the literature and is therefore presented here.

Numerical Solution of r = f(F)

The function of F for a given r can be expressed in two forms:

$$\frac{Z \exp(-Z)}{F(1 - \exp(-Z))} - r = 0 \quad \ldots\ldots\ldots\ldots \quad (2)$$

or

$$r = \frac{Z \exp(-Z)}{F(1 - \exp(-Z))} \quad \ldots\ldots\ldots\ldots \quad (3)$$

which gives

$$rF - ((rF+Z) \exp(-Z)) = 0 \quad \ldots\ldots\ldots\ldots \quad (4)$$

From the latter expression, the successive estimations (F_i) of the root F_t by Newton's method are:

$$F_{i+1} = F_i - \frac{f(F_i)}{f'(F_i)}$$

Thus

$$F_{i+1} = F_i - \frac{rF_i - (rF_i+Z_i) \exp(-Z_i)}{r + (rF_i+Z_i-r-1) \exp(-Z_i)}$$

or

$$F_{i+1} = \frac{(rF_i +Z_iF_i+M) \exp(-Z_i)}{r + (rF_i+Z-r-1) \exp(-Z_i)} \quad \ldots\ldots\ldots\ldots \quad (5)$$

Expression (5) is very easy to program and compute on simple programmable pocket calculators. The successive estimate of F_i are calculated until they differ by less than a given value, usually 10^{-3} or 10^{-4}. The convergence is fairly rapid and F_t is obtained with excellent accuracy.

A problem may arise, however, as, for some very low values of initial F_i (the previous r being large), the method tends to converge on a trivial and absurd root, Z = 0 or F = -M. The existence of this root is artificially introduced by the use of expression (4) instead of (2) which would require a longer program that modifies completely the curve of the function, although both forms have the same roots.

Expression (2) is undetermined for F = 0 and F = -M, and, for positive values of F, it is represented by branches of hyperbolas asymptotic to the line Y = r when F is large, and tend to be negative when F tends to 0. There is a single positive root and no negative root.

Expression (4) is represented by parabola-like curves, with concavity upwards and asymmetrical branches that all intercept the F-axis at F = -M and the ordinate axis at y = -M exp(-M). The smaller that r is, the wider the curve is "open" towards positive F and the larger the positive root F. Thus, if both r and the previous F are small, the tangent line to the curve at the corresponding point will intercept the F-axis on the negative size, so that the root will readily converge on $F = -M$.

This problem is very easy to eliminate in practice, as the function only has two roots with opposite signs, and the wrong root is immediately noticeable. When it appears, a larger initial F_i (1 or 2 in general) is introduced and the iteration is performed again for the same value of r.

Two solutions can be included in the program: either a large initial F is systematically entered to start all of the iterations, or a test of the sign of F is included in the program to eliminate the negative root automatically and return to the iterative step.

Description of the Program

The computation is performed through five parts.

1) Initialization. A starting value of F_T is assumed and the starting population size (N_T) is calculated with two possible formula: (i) $N_T = \frac{C_T\ Z_T}{F_T}$ if there is no survival after period T; and (ii) $N_T = \frac{C_T\ Z_T}{F_T(1-\exp(-Z_T))}$ if the fishing is incomplete and some individuals survive. N_T is stored and displayed.

2) Calculation of r. N_T and C_{T-1}, and later every couple (N_{t+1}, C_t) are recalled or entered; their ratio is calculated at the first step and stored.

3) Determination of the root (F_t). The corresponding root of expression (4) is determined iteratively from a starting value of F_i (i.e. F_T for the first computation) and then the successive values of F_t. This part proceeds with the use of expression (5), and is stopped when successive estimations of F_i are within the required precision range.

4) Elimination of the wrong root. A test is made on the sign of the resulting F_t. If it is positive, it is retained for the next step; if it is negative ($F_t = -M$), step (3) is performed again after a larger value of initial F_i has been entered.

5) Calculation of population size. N_{t+1} is recalled and multiplied by the reciprocal of the rate of survival ($N_t = N_{t+1} \exp(F_t+M)$). The program returns to step (2), with N_t displayed. C_{t-1} is entered and steps (2) to (5) are performed backwards to $t = t_c$.

A typical listing of this program for HP programmable pocket calculators is presented in Table 1. Depending on the model of calculator used, all or parts of the program may be performed automatically.

With the model HP-25 calculator used by the author, parts (2), (3) and (5) are performed with the program, while parts (1) and (4) must be done manually, but the whole computation is possible in a single program with models HP-29 or HP-67. The program occupies a small number of program steps and a small amount of memory.

In conclusion, NEWTON's method is shown to be quite accurate for the determination of the roots of the catch at age equation. The form used in the computation introduces a negative root that is easily eliminated. The full range of values of r can be entered and solved, as all the curves have the same general shape. There is no limit to the values of F and M that can be used.

References

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Table 1. Program for the solution of the catch equation with HP pocket calculators, using NEWTON's method.

Step	Key	Comments
	$N_{t=1}$	Displayed
00	↑	
	C_t	Entered
01	÷	
	STO 2	r
03	RCL 1	F_i
	RCL 2	
	×	
	STO 7	rF_i
	RCL 1	F_i
	RCL 0	M
	+	
10	STO 6	Z_i
	+	
	STO 4	rF + Z
	RCL 1	
	×	
	RCL 0	
	+	
	RCL 6	
	CHS	
20	e^x	numerator
	×	
	RCL 4	
	RCL 2	
	−	
	1	
	−	
	RCL 6	
	CHS	
	e^x	
	×	
30	RCL 2	
	+	Denominator
	÷	F_{i+1}
	RCL 1	
	↔	
	STO 1	
	−	
37	ABS	
38	RCL 5	EEX − n
	X ≥ Y	

Step	Key	Comments
40	GTO 42	
	GTO 03	
42	RCL 1	F_{i+1}
	X ≥ 0	Test of sign
	GTO 48	
	1	1 or 2
	STO 1	
	GTO 03	
48	R/S	Read F_t
	RCL 0	M
	+	
	e^x	1/S
	STO × 3	
	RCL 3	N_t
54	GTO 00	Return
		* Initialization
		Enter C_T
55	RCL 0	M
	RCL 1	F_T
	+	
	STO 6	Z_T
	×	
60	RCL 1	
	÷	
	STO 3	
63	R/S	N_T, fishing complete
	RCL 6	
	CHS	
	e^x	
	CHS	
	1	
	+	
	÷	
	STO 3	
72	GTO 00	N_T, fishing incomplete
		Return